Math 245 Spring 2013 Midterm Solutions

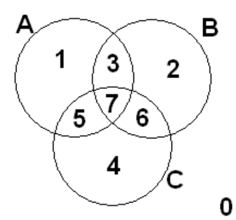
1. Find a formula involving the connectives $\vee, \wedge$, and $\neg$ that has this truth table:

P	Q	???	$P \wedge Q$	$\neg(P \wedge Q)$	$P \vee Q$	$(\neg(P \wedge Q)) \wedge (P \vee Q)$
F	F	F	F	T	F	F
F	T	T	F	T	T	T
T	F	T	F	T	T	T
T	T	F	T	F	T	F

Answer: $(\neg(P \wedge Q)) \wedge (P \vee Q)$. Other answers are possible.

2. What can we put in the blank to make the identity correct?
$(A \triangle B) \cap C = (C \setminus A) \triangle$ __________

Answer: $C \setminus B$. The simplest justification is by Venn diagram. $A \triangle B$ is the regions $1, 2, 5, 6$; intersecting with C gives the regions $5, 6$. $C \setminus A$ is the regions $4, 6$, while $C \setminus B$ is the regions $4, 5$. Taking the symmetric difference gives the regions $5, 6$.



3. Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \leftrightarrow Q$.

Answer: $\neg((P \rightarrow Q) \rightarrow \neg(Q \rightarrow P))$. This is equivalent to $\neg(\neg(P \rightarrow Q) \vee \neg(Q \rightarrow P))$ by the conditional law, which is equivalent to $\neg\neg(P \rightarrow Q) \wedge \neg\neg(Q \rightarrow P)$ by the second DeMorgan's law, which is equivalent to $(P \rightarrow Q) \wedge (Q \rightarrow P)$ by the double negation law (twice), which is equivalent to $P \leftrightarrow Q$ by the definition of biconditional.

4. Determine whether or not the following statements are equivalent: $(\exists x \in A\ P(x)) \wedge (\exists x \in B\ P(x))$ and $\exists x \in (A \cap B)\ P(x)$.

Answer: no. Here is a counterexample: let $A = \{2, 4\}, B = \{3\}$, and let $P(x)$ stand for the sentence "x is prime". A, B each contains a prime, but $A \cap B$ is empty, so does not contain a prime.

5. Prove that $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$.

$\subseteq$: Let $x \in \mathscr{P}(A \cap B)$. Then $x \subseteq A \cap B$. For all $y \in x$, $y \in A \cap B$. In particular, $\forall y \in x, y \in A$ (thus $x \subseteq A$) and $\forall y \in x, y \in B$ (thus $x \subseteq B$). Because $x \subseteq A$, $x \in \mathscr{P}(A)$; because $x \subseteq B$, $x \in \mathscr{P}(B)$. Combining these we get $x \in \mathscr{P}(A) \cap \mathscr{P}(B)$.

$\supseteq$: Let $x \in \mathscr{P}(A) \cap \mathscr{P}(B)$. Then $x \in \mathscr{P}(A)$ and $x \in \mathscr{P}(B)$, so $x \subseteq A$ and $x \subseteq B$. For all $y \in x$, $y \in A$ and $y \in B$, so $y \in A \cap B$. Hence $x \subseteq A \cap B$ and thus $x \in \mathscr{P}(A \cap B)$.

6. Suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

   Suppose that $A \setminus B \subseteq C \cap D$, $x \in A$, and $x \notin B$. Combining $x \in A$ and $x \notin B$ we get $x \in A \setminus B$. Because $A \setminus B \subseteq C \cap D$ we get $x \in C \cap D$, and in particular $x \in D$. We have proved that $x \notin B$ implies $x \in D$; the contrapositive of this is the desired goal.

7. Suppose that $x, y \in \mathbb{R}$. Prove that if $x \neq 0$, then if $y = \frac{3x^2+2y}{x^2+2}$ then $y = 3$.

   Suppose that $x \neq 0$ and $y = \frac{3x^2+2y}{x^2+2}$. Multiplying by the nonzero $x^2 + 2$ we get $yx^2 + 2y = 3x^2 + 2y$. Subtracting $2y$ we get $yx^2 = 3x^2$. Dividing by the nonzero x^2 we get $y = 3$, as desired.

8. Prove that if A and $B \setminus C$ are disjoint, then $A \cap B \subseteq C$.

   Suppose that A and $B \setminus C$ are disjoint. Let $x \in A \cap B$; hence $x \in A$ and $x \in B$. We argue by contradiction. Suppose that $x \notin C$. Combining with $x \in B$ we get $x \in B \setminus C$. But also $x \in A$; yet A and $B \setminus C$ are disjoint. This contradiction proves that $x \in C$. Since x was arbitrary in $A \cap B$, we have shown that $A \cap B \subseteq C$.

9. Prove that for every integer n, n^3 is even iff n is even.

   Let n be an integer. We proceed by cases, depending on if n is even or odd. If n is even, then for some integer m, $n = 2m$. Then $n^3 = (2m)^3 = 8m^3 = 2(4m^3)$, twice an integer, which is even. This proves that if n is even then n^3 is even.

   If n is odd, then for some integer k, $n = 2k + 1$. Then $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$, which is odd. This proves that if n is not even, then n^3 is not even.

10. Prove that for any sets A and B, if $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ then either $A \subseteq B$ or $B \subseteq A$.

    (method 1) Suppose that $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$. We argue by contradiction. Suppose that $\neg(A \subseteq B \vee B \subseteq A)$ holds. Hence $\neg(A \subseteq B) \wedge \neg(B \subseteq A)$ holds, by DeMorgan's second law. Hence $\neg(\forall x \in A\ x \in B) \wedge \neg(\forall y \in B\ y \in A)$. Hence $(\exists x \in A\ x \notin B) \wedge (\exists y \in B\ y \notin A)$. Now, consider the set $\{x, y\}$; let's name it z. Since $x, y \in A \cup B$, we have $z \subseteq A \cup B$ so $z \in \mathscr{P}(A \cup B)$. But $z \not\subseteq A$ since $y \notin A$; hence $z \notin \mathscr{P}(A)$. Also, $z \not\subseteq B$ since $x \notin B$; hence $z \notin \mathscr{P}(B)$. But this contradicts $z \in \mathscr{P}(A) \cup \mathscr{P}(B)$, which completes the proof.

    (method 2) Suppose that $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$. We argue by cases. Either $A \subseteq B$ or $A \not\subseteq B$. In the first case, we are done. In the second case, $\neg(A \subseteq B)$ holds, so $\neg(\forall x \in A\ x \in B)$. This implies that $\exists x \in A\ x \notin B$. Now, let $y \in B$. Consider the set $\{x, y\}$; let's name it z. Since $x, y \in A \cup B$, we have $z \subseteq A \cup B$ so $z \in \mathscr{P}(A \cup B)$. But $z \not\subseteq B$ since $x \notin B$; hence $z \notin \mathscr{P}(B)$. Because $z \in \mathscr{P}(A) \cup \mathscr{P}(B)$, in fact $z \in \mathscr{P}(A)$. Hence $y \in A$. Since $y \in B$ was arbitrary, we have proved that $B \subseteq A$.